81. Given, $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$

$\Rightarrow \quad 4a^2 f(-1) + 4a\ f(1) + f(2) = 3a^2 + 3a, \quad \dots \text{(i)}$

$4b^2 f(-1) + 4bf(1) + f(2) = 3b^2 + 3b \quad \dots \text{(ii)}$

And $4c^2 f(-1) + 4cf(1) + f(2) = 3c^2 + 3c \quad \dots. \text{(iii)}$

Where, $f(x)$ is quadratic expression given by, $f(x) = ax^2 + bx + c$ and equation (i), (ii) and (iii).

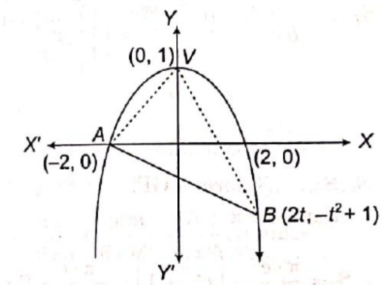
$\Rightarrow \quad 4x^2 f(-1) + 4x\ f(1) + f(2) = 3x^2 + 3x$ Or $\{4f(-1) - 3\}x^2 + \{4x\ f(1) - 3\} + f(2) = 0 \quad \dots \text{(iv)}$

As above equation has 3 roots a, b and c.
So, above equation is identity in x.
I.e. coefficients must be zero.

$\Rightarrow \quad f(-1) = \dfrac{3}{4},\ f(1) = \dfrac{3}{4},\ f(2) = 0 \quad \dots. \text{(iv)}$

$\because \quad f(x) = ax^2 + bx + c$

$\therefore \quad a = -\dfrac{1}{4}, b = 0$ and $c = 1$, using equation (v)



Thus, $f(x) = \dfrac{4 - x^2}{4}$ shown as,

Since AB subtends right angle at vertex $\therefore \quad t = 4$

$\therefore \quad B(8, -15)$

Let $A(-2, 0)$, $B = (2t, -t^2 + 1)$

Since, equation of chord AB is $y = \dfrac{-(3x+6)}{2}$.

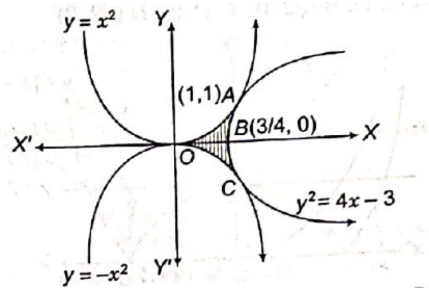
$$\therefore \quad \text{Required area} = \left| \int_{-2}^{8} \left(\frac{4 - x^2}{4} + \frac{3x + 6}{2} \right) dx \right| = \left| \left(x - \frac{x^3}{12} + \frac{3x^2}{4} + 3 \right)_{-2}^{8} \right|$$

$$= \left| \left[8 - \frac{128}{3} + 48 + 24 - \left(-2 + \frac{2}{3} + 3 - 6 \right) \right] \right| = \frac{125}{3} \text{ square units.}$$

82. The region bounded by the curves $y = x^2, y = -x^2$ and $y^2 = 4x - 3$ is symmetrical about X-axis, where $y = 4x - 3$ meets at (1, 1).

$\therefore \quad$ Area of curve (OABCO)

$$= 2\left[\int_0^1 x^2 dx - \int_{3/4}^{1} \left(\sqrt{4x - 3} \right) dx \right]$$



$$= 2\left[\left(\frac{x^3}{3} \right)_0^1 - \left(\frac{(4x-3)^{3/2}}{\frac{3 \cdot 4}{2}} \right)_{3/4}^{1} \right] = 2\left(\frac{1}{3} - \frac{1}{6} \right) = 1 \cdot \frac{1}{6} = \frac{1}{3} \text{ square units.}$$

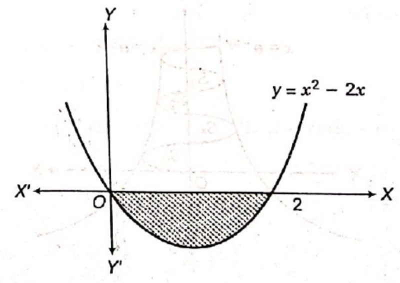
83. $\left(y = x^2 - 2x\ ,\ \frac{4}{3}\right)$

Here, slope of tangent, $\frac{dy}{dx} = \frac{(x+1)^2 + y - 3}{(x+1)}$ $\Rightarrow$ $\frac{dy}{dx} = (x+1) + \frac{(y-3)}{(x+1)}$,

Put $x + 1 = X$ and $y - 3 = Y$

$\Rightarrow$ $\frac{dy}{dx} = \frac{dY}{dX}$

$\therefore$ $\frac{dY}{dX} = X + \frac{Y}{X}$ $\Rightarrow$ $\frac{dY}{dX} - \frac{1}{X} Y = X$



If $= e^{\int -\frac{1}{X} dX} = e^{-\log X} = \frac{1}{X}$

$\therefore$ Solution is, $Y \cdot \frac{1}{X} = \int X \cdot \frac{1}{X} dX + c$ $\Rightarrow$ $\frac{Y}{X} = X + C$

$y - 3 = (x+1)^2 + c(x+1)$, which passes through (2, 0)

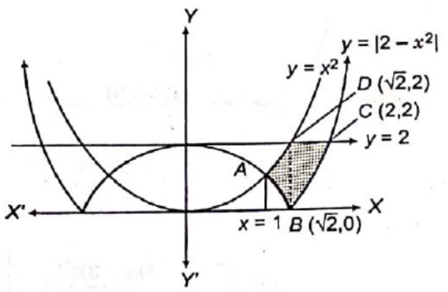
$\Rightarrow$ $-3 = (3)^2 + 3c$ $\Rightarrow$ $c = -4$

$\therefore$ Required curve $y = (x+1)^2 - 4(x+1) + 3$ $\Rightarrow$ $y = x^2 - 2x$

$\therefore$ Required area $= \left|\int_0^2 (x^2 - 2x)\, dx\right| = \left|\left(\frac{x^3}{3} - x^2\right)_0^2\right| = \frac{8}{3} - 4 = \frac{4}{3}$ square units.

84. The points is in the graph are

$A(1,1),\ B\left(\sqrt{2}, 0\right),\ C(2,2),\ D\left(\sqrt{2}, 2\right)$



$\therefore$ Required area $= \int_1^{\sqrt{2}} \{x^2 - (2 - x^2)\}\, dx + \int_{\sqrt{2}}^2 \{2 - (x^2 - 2)\}\, dx$

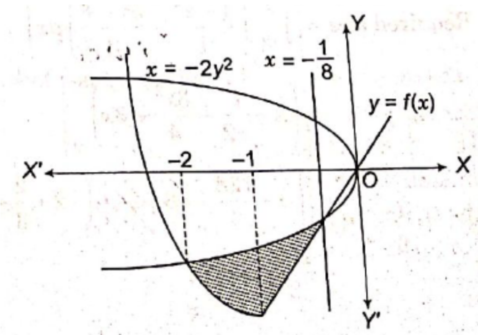
$= \int_1^{\sqrt{2}} (2x^2 - 2)\, dx + \int_{\sqrt{2}}^2 (4 - x^2)\, dx$

$= \left[\frac{2x^3}{3} - 2x\right]_1^{\sqrt{2}} + \left[4x - \frac{x^3}{3}\right]_{\sqrt{2}}^2$

$= \left[\frac{4\sqrt{2}}{3} - 2\sqrt{2} - \frac{2}{3} + 2\right] + \left[8 - \frac{8}{3} - 4\sqrt{2} + \frac{2\sqrt{2}}{3}\right] = \left(\frac{20 - 12\sqrt{2}}{3}\right)$ square units.

85. Given, $f(x) = \begin{cases} 2x & , |x| \le 1 \\ x^2 + ax + b & , |x| > 1 \end{cases}$

$\Rightarrow$ $f(x) = \begin{cases} x^2 + ax + b\ , & \text{if } x < -1 \\ 2x\ , & \text{if } -1 \le x < 1 \\ x^2 + ax + b\ , & \text{if } x \ge 1 \end{cases}$



f is continuous on R, so f is continuous at –1 and 1.

$\lim_{x \to 1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1)$

and $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

$\Rightarrow$ $1 - a + b = -2$ and $2 = 1 + a + b$

$\Rightarrow$ $a - b = 3$ and $a + b = 1$

$\therefore \quad a = 2, \quad b = -1$

Hence, $f(x) = \begin{cases} x^2 + 2x - 1 & , \text{ if } x < -1 \\ 2x & , -1 \le x < 1 \\ x^2 + 2x - 1 & , \quad x \ge 1 \end{cases}$

Next, we have to find the points $x = -2y^2$ and $y = f(x)$.

The point of intersection is (–2, –1).

$\therefore \quad$ Required area $= \int_{-2}^{-1/8} \left[\sqrt{\frac{-x}{2}} - f(x) \right] dx$

$$= \int_{-2}^{-1/8} \sqrt{\frac{-x}{2}}\, dx - \int_{-2}^{-1} (x^2 + 2x - 1)\, dx - \int_{-1}^{-1/8} 2x\, dx$$

$$= -\frac{2}{3\sqrt{2}} \left[(-x)^{3/2} \right]_{-2}^{-1/8} - \left[\left(\frac{x^3}{3} + x^2 - x \right) \right]_{-2}^{-1} - [x^2]_{-1}^{-1/8}$$

$$= -\frac{2}{3\sqrt{2}} \left[\left(\frac{1}{8} \right)^{3/2} - 2^{3/2} \right] - \left(-\frac{1}{3} + 1 + 1 \right) + \left(-\frac{8}{3} + 4 + 2 \right) - \left[\frac{1}{64} - 1 \right]$$

$$= \frac{\sqrt{2}}{3} [2\sqrt{2} - 2^{-9/2}] + \frac{5}{3} + \frac{63}{64} = \frac{63}{16 \times 3} + \frac{509}{64 \times 3} = \frac{761}{192} \text{ square units.}$$

86. Refer to the figure in the question. Let the coordinates of P be (x, x^2), where $0 \le x \le 1$.

For the area (OPRO), Upper boundary : $y = x^2$ and

Lower boundary : $y = f(x)$

Lower limit of x : 0

Upper limit of x : x

$\therefore \quad$ Area (OPRO) $= \int_0^x t^2\, dt - \int_0^x f(t)\, dt$

$$= \left[\frac{t^3}{3} \right]_0^x - \int_0^x f(t)\, dt = \frac{x^3}{3} - \int_0^x f(t)\, dt$$

For the area (OPQO),

The upper curve : $x = \sqrt{y}$ and the lower curve : $x = \frac{y}{2}$

Lower limit of y : 0

And upper limit of $y : x^2$

$\therefore \quad$ Area $(OPQO) = \int_0^{x^2} \sqrt{t}\, dt - \int_0^{x^2} \frac{t}{2}\, dt = \frac{2}{3}[t^{3/2}]_0^{x^2} - \frac{1}{4}[t^2]_0^{x^2} = \frac{2}{3}x^3 - \frac{x^4}{4}$

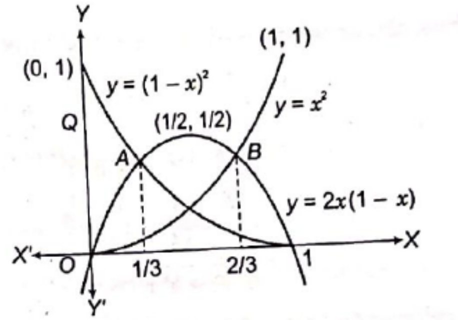
According the given condition, $\frac{x^3}{3} - \int_0^x f(t)\, dt = \frac{2}{3}x^3 - \frac{x^4}{4}$

On differentiating both sides w.r.t. x, we get

$x^2 - f(x) \cdot 1 = 2x^2 - x^3$

$\Rightarrow \quad f(x) = x^3 - x^2, \ 0 \le x \le 1$

87. We can drawn the graph of $y = x^2, y = (1 - x^2)$ and $y = 2x(1-x)$ in the following figure



Now, to get the point of intersection of $y = x^2$ and $y = 2x(1-x)$, we get $x^2 = 2x(1-x) \quad \Rightarrow \quad 3x^2 = 2x$

$\Rightarrow \quad x(3x-2) = 0 \quad \Rightarrow \quad x = 0, \dfrac{2}{3}$

Similarly, we can find the coordinate of the points of intersection of $y = (1-x^2)$ and $y = 2x(1-x)$ are $x = \dfrac{1}{3}$ and $x = 1$ from the figure, it is clear that, $f(x) = \begin{cases} (1-x)^2, & \text{if } 0 \le x \le \dfrac{1}{3} \\ 2x(1-x), & \text{if } \dfrac{1}{3} \le x \le \dfrac{2}{3} \\ x^2, & \text{if } \dfrac{2}{3} \le x \le 1 \end{cases}$

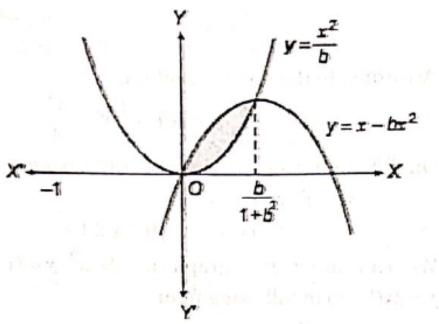
$\therefore$ The required area $A = \int_0^1 f(x)\,dx = \int_0^{1/3} (1-x)^2 dx + \int_{1/3}^{2/3} 2x(1-x)\,dx + \int_{2/3}^{1} x^2 dx$

$$= \left[-\frac{1}{3}(1-x)^3\right]_0^{1/3} + \left[x^2 - \frac{2x^3}{3}\right]_{1/3}^{2/3} + \left[\frac{1}{3}x^3\right]_{2/3}^{1}$$

$$= \left[-\frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{3}\right] + \left[\left(\frac{2}{3}\right)^2 - \frac{2}{3}\left(\frac{2}{3}\right)^3 - \left(\frac{1}{3}\right)^2 + \frac{2}{3}\left(\frac{1}{3}\right)^3\right] + \left[\frac{1}{3}(1) - \frac{1}{3}\left(\frac{2}{3}\right)^3\right]$$

$$= \frac{19}{81} + \frac{13}{81} + \frac{19}{81} = \frac{17}{27} \text{ square unit.}$$

88. Eliminating y from $y = \dfrac{x^2}{b}$ and $y = x - bx^2$, we get $x^2 = bx - b^2x^2 \quad \Rightarrow \quad x = 0, \dfrac{b}{1+b^2}$



Thus, the area enclosed between the parabolas $A = \int_0^{b/(1+b)^2} \left(x - bx^2 - \dfrac{x^2}{b}\right) dx = \left[\dfrac{x^2}{2} - \dfrac{x^3}{3} \cdot \dfrac{1+b^2}{b}\right]_0^{b/(1+b)^2}$

$$= \frac{1}{6} \cdot \frac{b^2}{(1+b^2)^2}$$

On differentiating w.r.t. b, we get

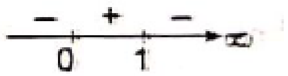
$$\frac{dA}{db} = \frac{1}{6} \cdot \frac{(1+b^2)^2 \cdot 2b - 2b^2 \cdot (1+b^2) \cdot 2b}{(1+b^2)^4} = \frac{1}{3} \cdot \frac{b(1-b^2)}{(1+b^2)^2}$$

For maximum value of A, put $\frac{dA}{db} = 0$

$\Rightarrow \quad b = -1, 0, 1$ since $b > 0$

$\therefore \quad$ We consider only $b = 1$.

Sign scheme for $\frac{dA}{db}$ around $b = 1$ is as shown below:



From sign scheme, it is clear that A is maximum at $b = 1$.

89. We have, $A_n = \int_0^{\pi/4} (\tan x)^n dx$

Since, $0 < \tan x < 1$, when $0 < x < \pi/4$

We have, $0 < (\tan x)^{n+1} < (\tan x)^n$ for each $n \in N$

$$\Rightarrow \quad \int_0^{\pi/4} (\tan x)^{n+1} dx < \int_0^{x^4} (\tan x)^n dx \quad \Rightarrow \quad A_{n+1} < A_n$$

Now, for $n > 2$

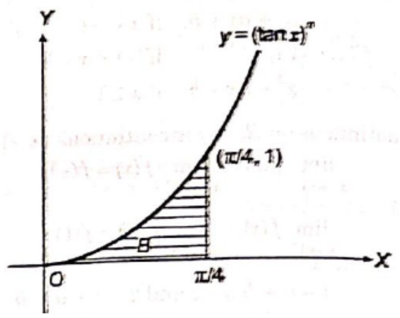
$$A_n + A_{n+2} = \int_0^{\pi/4} [(\tan x)^n + (\tan x)^{n+2}]\, dx = \int_0^{\pi/4} (\tan x)^n \,(1 + \tan^2 x) dx$$

$$= \int_0^{\pi/4} (\tan x)^n \sec^2 x \, dx = \left[\frac{1}{(n+1)} (\tan x)^{n+1}\right]_0^{\pi/4} = \frac{1}{(n+1)}(1-0) = \frac{1}{n+1}$$

Since, $A_{n+2} < A_{n+1} < A_n$

Then $A_n + A_{n+2} < 2\, A_n$

$$\Rightarrow \quad \frac{1}{n+1} < 2A_n \quad \Rightarrow \quad \frac{1}{2n+2} < A_n \quad \text{.... (i)}$$



Also, for $n > 2$

$$A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1} \quad \Rightarrow \quad 2A_n < \frac{1}{n-1}$$

$$\Rightarrow \quad A_n < \frac{1}{2n-2} \quad \text{... (ii)}$$

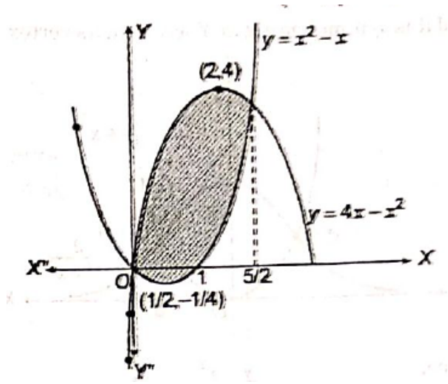
From equation (i) and (ii), $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$

90. Given parabolas are $y = 4x - x^2$ and $y = -(x-2)^2 + 4$ or $(x-2)^2 = -(y-4)$

Therefore, it is a vertically downwards parabola with vertex at (2, 4) and its axis is $x = 2$.

And $y = x^2 - x \quad \Rightarrow \quad y = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$

$\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 = y + \frac{1}{4}$



This is a parabola having its vertex at $\left(\frac{1}{2}, -\frac{1}{4}\right)$.

Its axis is at $x = \frac{1}{2}$ and opening upwards.

The points of intersection of given curves are

$4x - x^2 = x^2 - x \quad \Rightarrow \quad 2x^2 = 5x$

$\Rightarrow \quad x(2-5x) = 0 \quad \Rightarrow \quad x = 0, \frac{5}{2}$

Also, $y = x^2 - x$ meets X-axis at (0, 0) and (1, 0).

$\therefore$ Area, $A_1 = \int_0^{5/2}\left[\left(4x - x^2\right) - \left(x^2 - x\right)\right]dx = \int_0^{5/2}(5x - 2x^2)\,dx$

$= \left[\frac{5}{2}x^2 - \frac{2}{3}x^3\right]_0^{5/2} = \frac{5}{2}\left(\frac{5}{2}\right)^2 - \frac{2}{3}\cdot\left(\frac{5}{2}\right)^3 = \frac{5}{2}\cdot\frac{25}{4} - \frac{2}{3}\cdot\frac{125}{8}$

$= \frac{125}{8}\left(1 - \frac{2}{3}\right) = \frac{125}{24}$ square units.

This area is considering above and below X-axis both.

Now, for area below X-axis separately, we consider

$A_2 = -\int_0^1 (x^2 - x)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ square units.

Area above x axis = $A_1 - A_2 = \frac{121}{24}$

Hence, ratio of area above the X-axis and area below X-axis $= \frac{121}{24} : \frac{1}{6} = 121 : 4$